Journal of Mathematical Analysis and Applications **245**, 430–446 (2000)
doi:10.1006/jmaa.2000.6761, available online at http://www.idealibrary.com on IDEAL®

On Generalized Linear Singular Delay Systems

E. Grispos, G. Kalogeropoulos, and I. G. Stratis

University of Athens, Department of Mathematics, Panepistimiopolis

Communicated by C. T. Leondes

Received June 28, 1999

In this paper we study generalized linear singular delay systems. Such a system is reduced to a suitable homogeneous one. Using the Kronecker canonical form, the latter is decomposed into five subsystems, whose solutions are obtained. Moreover, the form of the initial function is given, so that the corresponding initial value problem is considered. © 2000 Academic Press

Key Words: matrix pencil theory; generalized singular linear delay systems.

0. INTRODUCTION

In many applications, one assumes that the system under consideration is governed by a principle of causality which intuitively requires that the cause precedes the effect; that is, the future state of the system is independent of the past states, and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state, then, generally, one is considering either ordinary or partial differential equations. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation, and that a more realistic model would include some of the past states of the system. Also, in some problems it is meaningless not to have dependence on the past. This has been known for some time, but the theory of such systems has been extensively developed only recently.

To the best of our knowledge, even in the case of the simplest type of past dependence in a differential equation (i.e., when this dependence is through the state variable and not its derivative), generalized systems have not been studied by the matrix pencil theory approach.

0022-247X/00 $35.00

This approach has been extensively used in control theory for the study of generalized linear time-invariant dynamical systems without delay. The first systematic study of matrix pencil theory is included in the book by Gantmacher [6]. Many researchers have implemented matrix pencil theory in their algebrogeometric consideration of a variety of problems. Closer to our spirit, we refer to the work of Campbell [2], Karcanias and Hayton [10], Van Dooren [5], and Kalogeropoulos [9].

The present paper may be considered as a continuation of [12], treating the case of generalized singular delay systems. It is organized as follows: in Section 1 are presented the necessary preliminary concepts from matrix pencil theory. Section 2 contains a brief account of the required elements of the theory of systems of linear delay differential equations. In Section 3 are developed the main results of this work. We start by observing that the study of a generalized linear singular delay system of the form

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t-r) + B\mathbf{u}(t), \qquad t > t_0, \qquad r > 0, \tag{0.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times \ell}$ ($\ell \leq n$), $\mathbf{u} \in C([t_0, \infty))$, may be reduced to studying an autonomous generalized linear delay system of the form

$$F\dot{\mathbf{x}}(t) = G\mathbf{x}(t-r) \tag{0.2}$$

under the assumption, usual in control theory, that rank $B = \ell$. When $sF - G$ is a singular pencil the system (0.2) is transformed, using the complex Kronecker form canonical decomposition of the pencil $sF - G$, into five easily solved subsystems. This procedure also suggests the form that the initial function should have, so that the corresponding (0.1) initial value problem admits a solution.

1. PRELIMINARY CONCEPTS FROM MATRIX PENCIL THEORY

Generalized linear systems of the form

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{1.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times l}$ are constant matrices, with det $E = 0$, and the input function $\mathbf{u}(t) \in \mathbb{C}^n$ is a vector function of a real variable t in time which is assumed to be sufficiently differentiable, are related to matrix pencil theory, since the algebraic, geometric, and dynamic properties stem from the structure by the associated pencil $sE - A$.

Through this paper we shall adopt the following notation: $\mathbb{R}, \mathbb{C}$ denote the field of real numbers and complex numbers, respectively. $\mathbb{N}$ is the set of natural numbers.

If $\mathbb{F}$ is a field, $\mathbb{F}^{m\times n}$ denotes the set of $m \times n$ matrices with elements from $\mathbb{F}$.

DEFINITION 1.1. Given $F, G \in \mathbb{R}^{m\times n}$ (or $\mathbb{C}^{m\times n}$), and an indeterminate s, then the matrix pencil $sF - G$ is said *singular* when $m \neq n$ or $m = n$ with $\det(sF - G)$ identically zero.

DEFINITION 1.2. The pencil $sF - G$ is said to be *strictly equivalent* to the pencil $sF_1 - G_1$ if and only if $P(sF - G)Q = sF_1 - G_1$, where $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, and $\det P, \det Q \neq 0$.

This strict equivalence relation can be defined rigorously on the set of singular pencils, denoted by $L^s_{m,n}$, as follows: We consider the set

$$g \coloneqq \left\{(P,Q) : P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}, \det P, \det Q \neq 0\right\} \tag{1.2}$$

and a composition rule $*$ defined on g as follows:

$$\begin{aligned} * : g \times g \to g : \left((P_1,Q_1),(P_2,Q_2)\right) &\to (P_1,Q_1)*(P_2,Q_2) \\ &\coloneqq (P_1 \cdot P_2, Q_2 \cdot Q_1). \end{aligned} \tag{1.3}$$

It may readily be verified that $(g, *)$ forms a non-abelian group. Furthermore, an action $\circ$ of the group $(g, *)$ on the set $L^s_{m,n}$ is defined by

$$\begin{aligned} \circ : g \times L^s_{m,n} \to L^s_{m,N} : \left((P,Q), sF - G\right) &\to (P,Q)\circ(sF - G) \\ &\coloneqq P(sF - G)Q. \end{aligned} \tag{1.4}$$

This group has the following properties:

(a) $(P_1, Q_1)\circ[(P_2, Q_2)\circ(sF - G)] = (P_1, Q_1)*(P_2, Q_2)\circ(sF - G)$, for every $P_1, P_2 \in \mathbb{C}^{m\times m}$, $Q_1, Q_2 \in \mathbb{C}^{n\times n}$, $\det P_1, \det P_2, \det Q_1, \det Q_2 \neq 0$, and $sF - G \in L^s_{m,n}$.

(b) $e_g \circ(sF - G) = sF - G$, $sF - G \in L^s_{m,n}$, where $e_g = (I_m, I_n)$ is the identity element of the group $(g, *)$. This action $\circ$ of the group $(g, *)$ on the set $L^s_{m,n}$ defines a transformation group denoted by M (see [13]). For $sF - G \in L^s_{m,n}$, the subset $g\circ(sF - G) \coloneqq \{(P,Q)\circ(sF - G) : (P,Q) \in g\} \subseteq L^s_{m,n}$ will be called the strict-equivalence relation and it is denoted by $\mathscr{E}_s$, so

$$(sF - G)\mathscr{E}_s(sF_1 - G_1) \qquad \text{if and only if } P(sF - G)Q = sF_1 - G_1,$$

where $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ $\det P, \det Q \neq 0$.

The class $\mathscr{E}_s(sF - G)$ is characterized by a uniquely defined element, known as the *complex Kronecker canonical form* $sF_k - G_k$ [6], specified by the complete set of invariants of $\mathscr{E}_s(sF - G)$.

Unlike the case of regular pencils, however, the characterization of the $\mathscr{E}_s(sF - G)$, $sF - G \in L^s_{m,n}$ apart from the set of elementary divisors requires the definition of additional sets of invariants, the minimal indices.

Thus assume that $m \neq n$ and that $r = \operatorname{rank}_{\mathbb{R}(s)}(sF - G) < \min\{m, n\}$, where $\mathbb{R}(s)$ denotes the field of rational functions in s, having real coefficients. (The notation of rank $(sF - G)$ has nothing to do with that of past dependence through the state variable in (0.1) and (0.2). It is simply a coincidence.)

Then the equations

$$(sF - G)\mathbf{x}(s) = \mathbf{0}, \qquad \boldsymbol{\psi}^t(s)(sF - G) = \mathbf{0}^t \tag{1.5}$$

have solutions in $\mathbf{x}(s)$ and $\boldsymbol{\psi}(s)$, which are vectors in the rational vector spaces $N_r(s) = N_r(sF - G)$ and $N_l(s) = N_l(sF - G)$, respectively, where

$$N_r(s) = \{\mathbf{x}(s) \in \mathbb{R}^n(s) : (sF - G)\mathbf{x}(s) = \mathbf{0}\}$$

and

$$N_l(s) = \{\boldsymbol{\psi}(s) \in \mathbb{R}^m(s) : \boldsymbol{\psi}^t(s)(sF - G) = \mathbf{0}^t\}.$$

Obviously $N_r(s)$ and $N_l(s)$ are vector spaces over $\mathbb{R}(s)$, with $\dim N_r(s) = n - r$ and $\dim N_l(s) = m - r$. It is known [14] that $N_r(s)$ and $N_l(s)$, as rational vector spaces, are spanned by minimal polynomial bases $\{\mathbf{x}_i(s), i = 1, 2, \dots, n - r\}$ and $\{\boldsymbol{\psi}_j^t(s), j = 1, 2, \dots, m - r\}$, correspondingly, of minimal degrees $\{\nu_1 = \nu_2 = \cdots = \nu_g = 0 < \nu_{g+1} \le \nu_{g+2} \le \cdots \le \nu_{n-r}\}$ and $\{\rho_1 = \rho_2 = \cdots = \rho_h = 0 < \rho_{h+1} \le \rho_{h+2} \le \cdots \le \rho_{m-r}\}$, respectively.

The sets of minimal degrees $\{\nu_i, 1 \le i \le n - r\}$ and $\{\rho_j, 1 \le j \le m - r\}$ are known [6] as *column minimal indices* (c.m.i.) and *row minimal indices* (r.m.i.) of $sF - G$, respectively. If $r = \operatorname{rank}_{\mathbb{R}(s)}(sF - G) < \min\{m, n\}$ it is evident that

$$r = \sum_{i=g+1}^{n-r} \nu_i + \sum_{j=h+1}^{m-r} \rho_j + \operatorname{rank}_{\mathbb{R}(s)}(sF_w - G_w),$$

where $sF_w - G_w$ is the complex Weierstrass canonical form [6], specified by the set of elementary divisors (e.d.) obtained by factorizing uniquely the invariant polynomials $f_i(s, \hat{s})$ over $\mathbb{R}[s, \hat{s}]$ (the ring of polynomials in s and $\hat{s} = 1/s$ with real coefficients), which are the non-zero elements of the diagonal of Smith canonical form of the homogeneous pencil $sF - \hat{s}G$, into powers of homogeneous polynomials irreducible over $\mathbb{C}$. These are of the following type:

$$s^d, \hat{s}^q \qquad \text{and} \qquad (s - a)^t, \qquad a \neq 0, \qquad d, q, t \in \mathbb{N}.$$

Let us consider that along with rank inequality we define the following types of invariants of $sF - G$:

- e.d. of the type s^d, $d \in \mathbb{N}$, are said zero elementary divisors (z.e.d.).
- e.d. of the type $(s-a)^t$, $a \neq 0$, $t \in \mathbb{N}$, are said non-zero finite elementary divisors (nz. f.e.d.).
- e.d. of the type $\hat{s}^q$ are said infinite elementary divisors (i.e.d.).
- c.m.i. of the type $\nu \in \mathbb{N} \cup \{0\}$ are said column minimal indices (c.m.i.) deduced from the column degrees of minimal polynomial bases of the maximal submodule M_N embedded in $N_r(s)$ with a free $\mathbb{R}(s)$-module structure.
- r.m.i. of the type $\rho \in \mathbb{N} \cup \{0\}$ are said row minimal indices (r.m.i.) deduced from the row degrees of minimal polynomial bases of the maximal submodule M_N embedded in $N_\ell(s)$ with a free $\mathbb{R}(s)$-module structure (see [9], [14]).

The complex Kronecker form $sF_k - G_k$ of the singular pencil $sF - G$ is defined

$$sF_k - G_k := \text{block diag}\left\{\mathbb{O}_{h,g}, s\Lambda_\nu - \lambda_\nu, s\Lambda_\rho^t - \lambda_\rho^t, sI_\tau - J_\tau, sH_q - I_q\right\},$$

where $\mathbb{O}_{h,g}$ is uniquely defined by the sets $\{\underbrace{0,0,\dots,0}_{g}\}$ and $\{\underbrace{0,0,\dots,0}_{h}\}$ of zero, column, and row minimal indices, respectively.

Also the second normal block $s\Lambda_\nu - \lambda_\nu$ is uniquely defined by the set of non-zero column minimal indices (a new arrangement of the indices of ν must be noted in order to simplify the notation) $\{\nu_1 \le \nu_2 \le \cdots \le \nu_{n-r-g}\}$ of $sF - G$ and has the form

$$s\Lambda_\nu - \lambda_\nu = \text{block diag}\left\{s\Lambda_{\nu_1} - \lambda_{\nu_1}, \dots, s\Lambda_{\nu_i} - \lambda_{\nu_i}, \dots, s\Lambda_{\nu_{n-r-g}} - \lambda_{\nu_{n-r-g}}\right\},$$

where $\Lambda_{\nu_i} = [I_{\nu_i} \vdots \mathbf{0}]$, $\lambda_{\nu_i} = [\mathbf{0} \vdots I_{\nu_i}]$ for every $i = 1, 2, \dots, n - r - g$, and I_{ν_i} and $\mathbf{0}$ denote the $\nu_i \times \nu_i$ identity matrix and the zero column matrix, respectively.

The third normal block $s\Lambda_\rho^t - \lambda_\rho^t$ is uniquely determined by the set of non-zero row minimal indices (a new arrangement of the indices of ρ must also be noted in order to simplify the notation) $\{\rho_1 \le \rho_2 \le \cdots \le \rho_{m-r-h}\}$ of $sF - G$ and has the form

$$s\Lambda_\rho^t - \lambda_\rho^t = \text{block diag}\left\{s\Lambda_{\rho_1}^t - \lambda_{\rho_1}^t, \dots, s\Lambda_{\rho_j}^t - \lambda_{\rho_j}^t, \dots, s\Lambda_{\rho_{m-r-h}}^t - \lambda_{\rho_{m-r-h}}^t\right\},$$

where

$$\Lambda_{\rho_j}^t = \begin{bmatrix} I_{\rho_j} \\ \cdots \\ \mathbf{0}^t \end{bmatrix}, \qquad \lambda_{\rho_j}^t = \begin{bmatrix} \mathbf{0}^t \\ \cdots \\ I_{\rho_j} \end{bmatrix},$$

for every $j = 1, 2, \ldots, m - r - h$, and I_{ρ_j} and $\mathbf{0}$ denote the $\rho_j \times \rho_j$ identity matrix and the zero row matrix, respectively.

The fourth normal Jordan type block $sI_\tau - J_\tau$ is uniquely defined by the set of f.e.d. $\{(s - a_1)^{\tau_1}, \ldots, (s - a_k)^{\tau_k}, \ldots, (s - a_\nu)^{\tau_\nu}\}$ $(\sum_{k=1}^{\nu} \tau_k = \tau)$ of $sF - G$ and has the form

$$sI_\tau - J_\tau = \text{block diag}\left\{sI_{\tau_1} - J_{\tau_1}(a_1), \ldots, sI_{\tau_k} - J_{\tau_k}(a_k), \ldots, sI_{\tau_\nu} - J_{\tau_\nu}(a_\nu)\right\},$$

where

$$J_{\tau_k}(a_k) = \begin{bmatrix} a_k & 1 & & & \\ & a_k & 1 & \mathbb{O} & \\ & & \ddots & \ddots & \\ & \mathbb{O} & & a_k & 1 \\ & & & & a_k \end{bmatrix}$$

is the $\tau_k \times \tau_k$ Jordan block at a_k.

Also the q block of the last normal block $sH_q - I_q$ is uniquely determined by the set of i.e.d. $\{(\hat{s})^{q_1}, \ldots, (\hat{s})^{q_\lambda}, \ldots, (\hat{s})^{q_\sigma}\}$ $(\sum_{\lambda=1}^{\sigma} q_\lambda = q,\ q_1 \leq q_2 \leq \cdots \leq q_\sigma)$ of $sF - G$ associated with the blocks $sH_{q_1} - I_{q_1}, \ldots, sH_{q_\lambda} - I_{q_\lambda}, \ldots, sH_{q_\sigma} - I_{q_\sigma}$. Then $sH_q - I_q = \text{block diag}\{sH_{q_1} - I_{q_1}, \ldots, sH_{q_\lambda} - I_{q_\lambda}, \ldots, sH_{q_\sigma} - I_{q_\sigma}\}$, and H_q is a nilpotent matrix of index $q^* = \max\{q_\lambda : \lambda = 1, 2, \ldots, \sigma\} = q_\sigma$, where $H_{q_\lambda} = J_{q_\lambda}(0)$.

The results of Section 1 are clarified by the following example:

Example. Suppose that the matrix pencil $sF - G \in L_{10,11}^s$ has the following set of invariants:

- f.e.d: $(s - 1)^2$,
- i.e.d: $(\hat{s})^2$,
- c.m.i: $0, 0, 2$,
- r.m.i: $0, 2$.

To each of these invariants, the corresponding block of the Kronecker canonical form is

$$(s-1)^2 \to sI_2 - J_2(1), \qquad (\hat{s})^2 \to sH_2 - I_2$$

$$0, 0 \text{ c.m.i and } 0 \text{ r.m.i} \to \mathbb{O}_{1,2} = [0 \ \ 0]$$

$$2 \text{ c.m.i} \to s\Lambda_2 - \lambda_2$$

$$= s\begin{bmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{bmatrix} - \begin{bmatrix} 0 & \vdots & 1 & 0 \\ 0 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & -1 & \vdots & 0 \\ 0 & s & \vdots & -1 \end{bmatrix}$$

$$2 \text{ r.m.i} \to s\Lambda_2^t - \lambda_2^t = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \cdots & \cdots \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 \\ -1 & s \\ \cdots & \cdots \\ 0 & -1 \end{bmatrix}.$$

Therefore we have

$$sF_k - G_k$$

$$= \begin{bmatrix} \mathbb{O}_{1,2} & & & & \\ & s\Lambda_2 - \lambda_2 & & & \\ & & s\Lambda_2^t - \lambda_2^t & & \\ & & & sI_\tau - J\tau & \\ & & & & sH_q - I_q \end{bmatrix}$$

$$= \begin{bmatrix} 0 \ \ 0 & & & & & & & & & \\ & s & -1 & 0 & & & & & & \\ & 0 & s & -1 & & & & & & \\ & & & & s & 0 & & & & \\ & & & & -1 & s & & & & \\ & & & & 0 & -1 & & & & \\ & & & & & & s-1 & -1 & & \\ & & & & & & 0 & s-1 & & \\ & & & & & & & & -1 & s \\ & & & & & & & & 0 & -1 \end{bmatrix},$$

where

$$sF_w - G_w = \begin{bmatrix} s-1 & -1 & & \\ 0 & s-1 & & \\ & & -1 & s \\ & & 0 & -1 \end{bmatrix}.$$

2. SYSTEMS OF LINEAR DELAY DIFFERENTIAL EQUATIONS

In this section we present in brevity the necessary main elements of the theory of systems of linear delay differential equations. The main result is

THEOREM 2.1. *Consider the system*

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t-r) + \mathbf{f}(t) \qquad t > t_0, \qquad r > 0 \tag{2.1}$$

and the initial condition

$$\mathbf{x}(t) = \phi(t), \qquad t_0 - r \le t \le t_0. \tag{2.2}$$

Let $A \in \mathbb{C}^{n\times n}$, $\mathbf{f} \in C([t_0,\infty))$, $\phi \in C([t_0 - r, t_0])$. *Then there exists a unique function*

$$\mathbf{x} \in C([t_0 - r, \infty)) \cap C^1((t_0,\infty))$$

that satisfies (2.1), (2.2).

Remark 2.1. (1) The proof may be found in [1, 3, 8].

(2) The function

$$\det(\lambda I - Ae^{-\lambda\tau})$$

is called the characteristic quasipolynomial of (2.1), while the equation

$$\det(\lambda I - Ae^{-\lambda\tau}) = 0 \tag{2.3}$$

is called the characteristic equation of (2.1). In general, (2.3) has infinitely many complex solutions λ.

(3) The superposition principle is valid; it extends to the case of a series of solutions provided it converges and admits term-by-term differentiation.

(4) Let (2.1) be written in the form

$$L\mathbf{x} = \mathbf{f} \tag{2.4}$$

and let $\mathbf{x}$ be a solution. Then $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ are solutions of the equations

$$L\mathbf{x} = \operatorname{Re}\mathbf{f} \qquad \text{and} \qquad L\mathbf{x} = \operatorname{Im}\mathbf{f},$$

respectively.

(5) To every root λ_j of (2.3) there corresponds a particular solution of the form

(a) λ_j: simple real root $\to e^{\lambda_j t}$

(b) λ_j: complex root ($\lambda_j = p_j \pm iq_j$) $\to e^{p_j t} \sin q_j t$, $e^{p_j t} \cos q_j t$

(c) λ_j: real root of multiplicity $a_j \to e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{a_j - 1} e^{\lambda_j t}$

(6) For a detailed study of the characteristic quasipolynomial, and the form of the solutions of (2.1), we refer to [1, 4].

Remark 2.2. In our case, all roots λ of (2.3) have negative real parts. Therefore, by [3, Sect. 28, Theorem B], if $\boldsymbol{\phi}$ is bounded, then the solution of $\{(2.1), (2.2)\}$ is bounded too.

Remark 2.3. We also have [3, 8] the variation of parameters method; let u be the unit step function on $[-r, 0]$

$$u(\theta) := \begin{cases} 0, & -r \leq \theta < 0 \\ 1, & \theta \geq 0. \end{cases}$$

Let, moreover, $\mathbf{x}_0(t; t_0, \boldsymbol{\phi})$ be the (unique) solution of the homogeneous problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t - r), \qquad t > t_0 \\ \mathbf{x}(t) &= \boldsymbol{\phi}(t), \qquad t_0 - r \leq t \leq t_0, \end{aligned} \tag{2.5}$$

with $\boldsymbol{\phi} \in C([t_0 - r, t_0])$.

Then, for $\mathbf{f} \in C([t_0, \infty))$, the non-homogeneous system $\{(2.1), (2.2)\}$ has a unique solution $\mathbf{x}(t)$ given by

$$\mathbf{x}(t) = \mathbf{x}_0(t; t_0, \boldsymbol{\phi}) + \int_{t_0}^{t} \mathbf{x}_0(t; s, \mathbf{f}(s)u)\, ds, \qquad t \geq t_0 - r.$$

Remark 2.4. (1) As far as the form of solutions is concerned, we refer to [1, Theorems 6.3, 6.4, 6.5, 6.6.]

(2) The asymptotic behavior of solutions is studied in [1, 8].

3. MAIN RESULTS

In this section, we deal with the initial value problem for generalized linear singular delay systems. These are systems of the form

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t - r) + B\mathbf{u}(t), \qquad t > t_0, \qquad r > 0 \tag{3.1}$$

$$\mathbf{x}(t) = \boldsymbol{\phi}(t), \qquad t_0 - r \leq t \leq t_0, \tag{3.2}$$

where $E, A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times \ell}$, the matrix pencil $sE - A$ is supposed to be regular, $\mathbf{u} \in C([t_0, \infty])$, and $\boldsymbol{\phi} \in C([t_0 - r, t_0])$.

LEMMA 3.1. *The system* (3.1) *may be represented as an autonomous generalized linear delay system*, *i.e.*, *in the form*

$$F\dot{\mathbf{x}}(t) = G\mathbf{x}(t-r). \tag{3.3}$$

Proof. Assume that rank $B = \ell < n$, and let $N \in \mathbb{C}^{(n-\ell)\times n}$, $M \in \mathbb{C}^{\ell\times n}$ be a left-annihilator and a left-inverse of B, respectively:

$$NB = \mathbb{O}_{n-\ell,\ell}, \qquad MB = I_\ell.$$

Then, if we set

$$L = \begin{bmatrix} N \\ \cdots \\ M \end{bmatrix} \in \mathbb{C}^{n\times n}, \tag{3.4}$$

we have [11] that $\det L \neq 0$.

Premultiplying (3.1) by L, we obtain the following, equivalent to (3.1), equations

$$NE\dot{\mathbf{x}}(t) = NA\mathbf{x}(t-r) \tag{3.5}$$

$$\mathbf{u}(t) = M[E\dot{\mathbf{x}}(t) - A\mathbf{x}(t-r)]. \tag{3.6}$$

Clearly, (3.5) is of the form (3.3). Moreover, for any solution $\mathbf{x}(t)$ of (3.5), the corresponding $\mathbf{u}(t)$ that generates $\mathbf{x}(t)$ is given by (3.6). ∎

Remark 3.1. An alternative approach for treating (3.1) is suggested by Remark 2.3, provided the solution of the corresponding homogeneous delay system is known.

In view of Lemma 3.1, we consider in the sequel systems of the form (3.3), where the corresponding matrix pencil $sF - G$ is singular.

From the singularity of $sF - G$, there exist non-singular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ such that (see also Section 1)

$$PFQ = F_k = \text{block diag}\left\{\mathbb{O}_{h,g}, \Lambda_\nu, \Lambda_\rho^t, I_\tau, H_q\right\} \tag{3.7}$$

$$PGQ = \text{block diag}\left\{\mathbb{O}_{h,g}, \lambda_\nu, \lambda_\rho^t, J_\tau, I_q\right\}. \tag{3.8}$$

THEOREM 3.1. *The system* (3.3) *may be decomposed in the equivalent set of subsystems*

$$\mathbb{O}_{h,g}\dot{\boldsymbol{\varphi}}_g(t) = \mathbb{O}_{h,g}\boldsymbol{\varphi}_g(t-r) \tag{3.9}$$

$$\Lambda_\nu\dot{\boldsymbol{\varphi}}_\nu(t) = \lambda_\nu\boldsymbol{\varphi}_\nu(t-r) \tag{3.10}$$

$$\Lambda_\rho^t\dot{\boldsymbol{\varphi}}_\rho(t) = \lambda_\rho^t\boldsymbol{\varphi}_\rho(t-r) \tag{3.11}$$

$$\dot{\boldsymbol{\varphi}}_\tau(t) = J_\tau\boldsymbol{\varphi}_\tau(t-r) \tag{3.12}$$

$$H_q\dot{\boldsymbol{\varphi}}_q(t) = \boldsymbol{\varphi}_q(t-r) \tag{3.13}$$

under a suitable transformation of $\mathbf{x}(t)$.

Proof. Consider the transformation

$$\mathbf{x}(t) = Q\boldsymbol{\psi}(t). \tag{3.14}$$

Under (3.14), system (3.3) becomes

$$FQ\dot{\boldsymbol{\psi}}(t) = GQ\boldsymbol{\psi}(t-r),$$

whereby, premultiplying by P, we arrive at

$$F_k\dot{\boldsymbol{\psi}}(t) = G_k\boldsymbol{\psi}(t-r). \tag{3.15}$$

By writing $\boldsymbol{\psi}(t) = [\boldsymbol{\varphi}_g^t(t), \boldsymbol{\varphi}_\nu^t(t), \boldsymbol{\varphi}_\rho^t(t), \boldsymbol{\varphi}_\tau^t(t), \boldsymbol{\varphi}_q^t(t)]^t$ and taking into account (3.7) and (3.8), we arrive at (3.9)–(3.13). ∎

In the sequel we study the initial value problems corresponding to the subsystems (3.9)–(3.13) taking into account that $\boldsymbol{\phi}(t)$ is written as

$$\boldsymbol{\phi}(t) = \left[\boldsymbol{\phi}_g^t(t), \boldsymbol{\phi}_\nu^t(t), \boldsymbol{\phi}_\rho^t(t), \boldsymbol{\phi}_\tau^t(t), \boldsymbol{\phi}_q^t(t)\right]^t.$$

PROPOSITION 3.1. *The initial value problem*

$$\begin{aligned} \mathbb{O}_{h,g}\dot{\boldsymbol{\varphi}}_g(t) &= \mathbb{O}_{h,g}\boldsymbol{\varphi}_g(t-r), && t > t_0, \quad r > 0 \\ \boldsymbol{\varphi}_g(t) &= \boldsymbol{\phi}_g(t), && t_0 - r \le t \le t_0 \end{aligned} \tag{3.9.1}$$

is satisfied for any initial column vector function $\boldsymbol{\phi}_g \in C([t_0 - r, t_0])$ *of* g *coordinates.*

Proof. The proof is obvious from the fact that left factors of $\dot{\boldsymbol{\varphi}}_g$ and $\boldsymbol{\varphi}_g$ are the $h \times g$ zero matrices. ∎

PROPOSITION 3.2. *Let* $\nu_i \in \mathbb{N}$ *be a non-zero column minimal index of the pencil* $sF - G \in L_{m,n}^s$. *Moreover, let the corresponding typical initial value problem from* (3.10) *be*

$$\begin{aligned} \Lambda_{\nu_i}\dot{\boldsymbol{\varphi}}_{\nu_i+1}(t) &= \lambda_{\nu_i}\boldsymbol{\varphi}_{\nu_i+1}(t-r), && t > t_0, \quad r > 0 \\ \boldsymbol{\varphi}_{\nu_i+1}(t) &= \boldsymbol{\phi}_{\nu_i+1}(t), && t_0 - r \le t \le t_0, \end{aligned} \tag{3.10.1}$$

with index i *taking a value between* 1 *and* $n - r - g$.

If $\boldsymbol{\varphi}_{\nu_i+1}(t) := [\varphi_1(t) \;\; \varphi_2(t) \;\; \cdots \;\; \varphi_{\nu_i}(t) \;\; \varphi_{\nu_i+1}(t)]^t$ *and*

$$\phi_{\nu_i+1}(t) := \left[\phi_1(t) \;\; \phi_2(t) \;\; \cdots \;\; \phi_{\nu_i}(t) \;\; \phi_{\nu_i+1}(t)\right]^t,$$

an initial function are considered, where $\varphi_{\nu_i+1}\colon [t_0, \infty) \to \mathbb{R}$ *is an arbitrary* ν_i*-times integrable function over* $[t_0, \infty)$ *with*

$$\boldsymbol{\varphi}_{\nu_i+1}(t) = \left[\underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{t}}_{\nu_i} \varphi_{\nu_i+1}(s)\,ds \;\cdots\; \int_{t_0}^{t} \varphi_{\nu_i+1}(s)\,ds \quad \varphi_{\nu_i+1}(t)\right]^t$$

and $\phi_1, \phi_2(t), \ldots, \phi_{\nu_i}, \phi_{\nu_i+1} \in C([t_0 - r, t_0])$ *such that*

$$\mathbf{w}_{\nu_i+1}(t) := \begin{cases} \boldsymbol{\varphi}_{\nu_i+1}(t), & t > t_0 \\ \boldsymbol{\phi}_{\nu_i+1}(t), & t_0 - r \le t \le t_0, \end{cases}$$

with $\mathbf{w}_{\nu_i+1}(t) := [w_1(t)\ \ w_2(t)\ \ \cdots\ \ w_{\nu_i}(t)\ \ w_{\nu_i+1}(t)]^t$ *and* $w_1, w_2, \ldots, w_{\nu_i}, w_{\nu_i+1} \in C([t_0 - r, \infty])$, *then* $\mathbf{w}_{\nu_i+1}$ *satisfies the initial value problem*

$$\begin{aligned} \Lambda_{\nu_i}\dot{\mathbf{w}}_{\nu_i+1}(t) &= \lambda_{\nu_i}\mathbf{w}_{\nu_i+1}(t-r), && t > t_0 \\ \mathbf{w}_{\nu_i+1}(t) &= \phi_{\nu_i+1}(t), && t_0 - r \le t \le t_0. \end{aligned} \tag{3.10.2}$$

Proof. By the definition of Λ_{ν_i} and λ_{ν_i} it follows that the first of system (3.10.2) can be written as

$$\begin{bmatrix} I_{\nu_i} & \vdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{w}_1(t) \\ \vdots \\ \dot{w}_{\nu_i+1}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \vdots & I_{\nu_i} \end{bmatrix} \begin{bmatrix} w_1(t-r) \\ \vdots \\ w_{\nu_i+1}(t-r) \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} \dot{w}_1(t) \\ \vdots \\ \dot{w}_{\nu_i}(t) \end{bmatrix} = \begin{bmatrix} w_2(t-r) \\ \vdots \\ w_{\nu_i+1}(t-r) \end{bmatrix}. \tag{3.10.3}$$

Such a system is always consistent. If we take $w_{\nu_i+1}(t)$ to be an arbitrary ν_i-times integrable function, then all w_k, $k = 1, 2, \ldots, \nu_i$ may be determined by successive integrations of $w_{\nu_i+1}(t)$ from (3.10.3). It is then clear that $\mathbf{w}_{\nu_i+1}(t)$ satisfies the initial value problem (3.10.2). ∎

Remark 3.2. Since the index i runs through all values from 1 up to $n - r - g$, the subsystem (3.10) of (3.3) considered for the solution with initial condition $\boldsymbol{\varphi}_\nu(t) = \boldsymbol{\phi}_\nu(t)$, $t_0 - r \le t \le t_0$, has the extended form (3.10.1) for every $1 \le i \le n - r - g$. From the proof of Proposition 3.2 for a specific value of i we take the solutions of (3.10) with initial condition $\boldsymbol{\varphi}_\nu(t) = \boldsymbol{\phi}_\nu(t)$, $t_0 - r \le t \le t_0$.

So let $\nu_1, \nu_2, \ldots, \nu_{n-r-g} \in \mathbb{N}$ be all non-zero column minimal indices of the pencil $sF - G \in L^s_{m,n}$ and the corresponding initial value problem

$$\begin{aligned} \Lambda_\nu \dot{\boldsymbol{\varphi}}_\nu(t) &= \lambda_\nu \boldsymbol{\varphi}_\nu(t-r), && t > t_0 \\ \boldsymbol{\varphi}_\nu(t) &= \boldsymbol{\phi}_\nu(t), && t_0 - r \le t \le t_0. \end{aligned} \tag{3.10.4}$$

If it is denoted

$$\boldsymbol{\varphi}_\nu(t) = \Big[\varphi_1(t) \cdots \varphi_{\nu_1+1}(t) \vdots \varphi_{\nu_1+2}(t) \cdots \varphi_{\nu_1+\nu_2+2}(t) \vdots \cdots \vdots$$
$$\varphi_{\nu_1+\nu_2+\cdots+\nu_{\mathbf{i}-1}+i}(t) \cdots \varphi_{\nu_1+\nu_2+\cdots+\nu_{\mathbf{i}}+i}(t) \vdots \cdots \vdots$$
$$\varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g-1}+n-r-g}(t) \cdots \varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g}+n-r-g}(t)\Big]^t$$

and the initial vector $\boldsymbol{\phi}_\nu(t)$ follows the composite form of $\boldsymbol{\varphi}_\nu$ with coordinates $\varphi_\beta(t)$, where $\beta = 1, 2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{n-r-g} + n - r - g$, by considering $\varphi_{\nu_1+1}, \varphi_{\nu_1+\nu_2+2}, \ldots, \varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g}+n-r-g}\colon [t_0, \infty) \to \mathbb{R}$, arbitrary $\nu_1, \nu_2, \ldots, \nu_{n-r-g}$-times integrable functions, respectively, over $[t_0, \infty)$ with

$$\boldsymbol{\varphi}_\nu(t) = \Bigg[\underbrace{\int_{t_0}^t \cdots \int_{t_0}^t}_{\nu_1} \varphi_{\nu_1+1}(s)\,ds \cdots \int_{t_0}^t \varphi_{\nu_1+1}(s)\,ds \quad \varphi_{\nu_1+1}(t) \vdots$$
$$\underbrace{\int_{t_0}^t \cdots \int_{t_0}^t}_{\nu_2} \varphi_{\nu_1+\nu_2+2}(s)\,ds \cdots \int_{t_0}^t \varphi_{\nu_1+\nu_2+2}(s)\,ds \quad \varphi_{\nu_1+\nu_2+2}(t) \vdots \cdots \vdots$$
$$\underbrace{\int_{t_0}^t \cdots \int_{t_0}^t}_{\nu_i} \varphi_{\nu_1+\nu_2+\cdots+\nu_i+i}(s)\,ds \cdots$$
$$\int_{t_0}^t \varphi_{\nu_1+\nu_2+\cdots+\nu_{\mathbf{i}}+i}(s)\,ds \quad \varphi_{\nu_1+\nu_2+\cdots+\nu_{\mathbf{i}}+i}(t) \vdots \cdots \vdots$$
$$\underbrace{\int_{t_0}^t \cdots \int_{t_0}^t}_{\nu_{n-r-g}} \varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g}+n-r-g}(s)\,ds \cdots$$
$$\int_{t_0}^t \varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g}+n-r-g}(s)\,ds \quad \varphi_{\nu_1+\nu_2+\cdots+\nu_{n-r-g}+n-r-g}(t)\Bigg]^t$$

and $\phi_\beta(t) \in C([t_0 - r, t_0])$ such that

$$\mathbf{w}_\nu(t) := \begin{cases} \boldsymbol{\varphi}_\nu(t), & t > t_0 \\ \boldsymbol{\phi}_\nu(t), & t_0 - r \le t \le t_0, \end{cases}$$

which follows the composite form of $\boldsymbol{\varphi}_\nu$ and $\boldsymbol{\phi}_\nu$ with coordinates $w_\beta(t) \in C([t_0 - r, \infty))$ for every $\beta = 1, 2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{n-r-g} + n - r - g$, then $\mathbf{w}_\nu(t)$ satisfies the initial value problem

$$\begin{aligned} \Lambda_\nu \dot{\mathbf{w}}_\nu(t) &= \lambda_\nu \mathbf{w}_\nu(t - r), && t > t_0 \\ \mathbf{w}_\nu(t) &= \boldsymbol{\phi}_\nu(t), && t_0 - r \le t \le t_0. \end{aligned} \tag{3.10.5}$$

PROPOSITION 3.3. *Let* $\rho_j \in \mathbb{N}$ *be a non-zero row minimal index of the pencil* $sF - G \in L^s_{m,n}$. *Moreover, let the corresponding typical initial value problem from* (3.11) *be*

$$\begin{aligned} \Lambda^t_{\rho_j} \dot{\boldsymbol{\varphi}}_{\rho_j}(t) &= \lambda^t_{\rho_j} \boldsymbol{\varphi}_{\rho_j}(t-r), \qquad t > t_0, \qquad r > 0 \\ \boldsymbol{\varphi}_{\rho_j}(t) &= \boldsymbol{\phi}_{\rho_j}(t), \qquad t_0 - r \le t \le t_0, \end{aligned} \tag{3.11.1}$$

with index j taking a value between 1 *and* $m - r - h$.

If $\boldsymbol{\varphi}_{\rho_j}(t) := [\tilde{\varphi}_1(t) \ \ \tilde{\varphi}_2(t) \ \ \cdots \ \ \tilde{\varphi}_{\rho_j - 1}(t) \ \ \tilde{\varphi}_{\rho_j}(t)]^t$ *and*

$$\boldsymbol{\phi}_{\rho_j}(t) := \left[\tilde{\phi}_1(t) \ \ \tilde{\phi}_2(t) \ \ \cdots \ \ \tilde{\phi}_{\rho_j - 1}(t) \ \ \tilde{\phi}_{\rho_j}(t) \right]^t,$$

an initial function, are considered, where $\tilde{\phi}_x \in C([t_0 - r, t_0])$ *for every* $x = 1, 2, \ldots, \rho_j$, *with* $\tilde{\phi}_x(t_0) = 0$ *for every* $x = 1, 2, \ldots, \rho_j$, *then the function*

$$\mathbf{w}_{\rho_j}(t) := \begin{cases} \mathbf{0}^t \in \mathbb{R}^{\rho_j}, & t > t_0 \\ \boldsymbol{\phi}_{\rho_j}(t), & t_0 - r \le t \le t_0, \end{cases}$$

with $\mathbf{w}_{\rho_j}(t) := [\tilde{w}_1(t) \ \ \tilde{w}_2(t) \ \ \cdots \ \ \tilde{w}_{\rho_j - 1}(t) \ \ \tilde{w}_{\rho_j}(t)]^t$, *satisfies*

$$\begin{aligned} \Lambda^t_{\rho_j} \dot{\mathbf{w}}_{\rho_j}(t) &= \lambda^t_{\rho_j} \mathbf{w}_{\rho_j}(t-r), \qquad t > t_0 \\ \mathbf{w}_{\rho_j}(t) &= \boldsymbol{\phi}_{\rho_j}(t), \qquad t_0 - r \le t \le t_0. \end{aligned} \tag{3.11.2}$$

Proof. By the definition of $\Lambda^t_{\rho_j}$ and $\lambda^t_{\rho_j}$, it follows that the system (3.11.2) may be written in the form

$$\begin{bmatrix} I_{\rho_j} \\ \cdots \\ \mathbf{0}^t \end{bmatrix} \begin{bmatrix} \dot{\tilde{w}}_1(t) \\ \vdots \\ \dot{\tilde{w}}_{\rho_j}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}^t \\ \cdots \\ I_{\rho_j} \end{bmatrix} \begin{bmatrix} \tilde{w}_1(t-r) \\ \vdots \\ \tilde{w}_{\rho_j}(t-r) \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} \dot{\tilde{w}}_1(t) \\ \vdots \\ \dot{\tilde{w}}_{\rho_j}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{w}_1(t-r) \\ \vdots \\ \tilde{w}_{\rho_j}(t-r) \end{bmatrix}, \tag{3.11.3}$$

whereby we have that $\mathbf{w}_{\rho_j}(t) = \mathbf{0}$, $t \ge t_0$. The result follows by the assumption on the initial value function $\boldsymbol{\phi}_{\rho_j}(t)$. ∎

Remark 3.3. Since the index j runs through all values from 1 up to $m-r-h$, the subsystem (3.11) of (3.3) considered for the solution with initial condition $\boldsymbol{\varphi}_\rho(t) = \boldsymbol{\phi}_\rho(t)$, $t_0 - r \le t \le t_0$, has the extended form (3.11.1) for every $1 \le j \le m-r-h$. From the proof of Proposition 3.3 for a specific value of j and following thinking similar to Remark 3.2 we come to the following result.

Let $\rho_1, \rho_2, \dots, \rho_{m-r-h} \in \mathbb{N}$ be all non-zero row minimal indices of the pencil $sF - G \in L^s_{m,n}$ and the corresponding initial value problem

$$\begin{aligned} \Lambda^t_\rho \dot{\boldsymbol{\varphi}}_\rho(t) &= \lambda^t_\rho \boldsymbol{\varphi}_\rho(t-r), && t > t_0 \\ \boldsymbol{\varphi}_\rho(t) &= \boldsymbol{\phi}_\rho(t), && t_0 - r \le t \le t_0. \end{aligned} \tag{3.11.4}$$

If

$$\begin{aligned} \boldsymbol{\varphi}_\rho(t) = \Big[& \tilde{\varphi}_1(t) \cdots \tilde{\varphi}_{\rho_1}(t) \vdots \tilde{\varphi}_{\rho_1+1}(t) \cdots \tilde{\varphi}_{\rho_1+\rho_2}(t) \vdots \cdots \vdots \\ & \tilde{\varphi}_{\rho_1+\rho_2+\cdots+\rho_{j-1}+1}(t) \cdots \tilde{\varphi}_{\rho_1+\rho_2+\cdots+\rho_j}(t) \vdots \cdots \vdots \\ & \tilde{\varphi}_{\rho_1+\varphi_2+\cdots+\rho_{m-r-h-1}+1}(t) \cdots \tilde{\varphi}_{\rho_1+\rho_2+\cdots+\rho_{m-r-h}}(t) \Big]^t \end{aligned}$$

and the initial vector $\boldsymbol{\phi}_\rho(t)$ follows the composite form of $\boldsymbol{\varphi}_\rho$ with coordinates $\tilde{\varphi}_\gamma(t) \in C([t_0 - r, t_0])$, for every $\gamma = 1, 2, \dots, \rho_1 + \rho_2 + \cdots + \rho_{m-r-h}$, by considering $\tilde{\varphi}_\gamma(t_0) = 0$ for every $\gamma = 1, 2, \dots, \rho_1 + \rho_2 + \cdots + \rho_{m-r-h}$, then the function

$$\mathbf{w}_\rho(t) := \begin{cases} \mathbf{0}^t \in \mathbb{R}^{\rho_1+\rho_2+\cdots+\rho_{m-r-h}}, & t > t_0 \\ \boldsymbol{\phi}_\rho(t), & t_0 - r \le t \le t_0 \end{cases}$$

with

$$\begin{aligned} \mathbf{w}_\rho(t) = \Big[& \tilde{w}_1(t) \cdots \tilde{w}_{\rho_1}(t) \vdots \tilde{w}_{\rho_1+1}(t) \cdots \tilde{w}_{\rho_1+\rho_2}(t) \vdots \cdots \vdots \\ & \tilde{w}_{\rho_1+\rho_2+\cdots+\rho_{j-1}+1}(t) \cdots \tilde{w}_{\rho_1+\rho_2+\cdots+\rho_j}(t) \vdots \cdots \vdots \\ & \tilde{w}_{\rho_1+\rho_2+\cdots+\rho_{m-r-h-1}+1}(t) \cdots \tilde{w}_{\rho_1+\rho_2+\cdots+\rho_{m-r-h}}(t) \Big]^t \end{aligned}$$

satisfies the initial value problem

$$\begin{aligned} \Lambda^t_\rho \dot{\mathbf{w}}_\rho(t) &= \lambda^t_\rho \mathbf{w}_\rho(t-r), && t > t_0 \\ \mathbf{w}_\rho(t) &= \boldsymbol{\phi}_\rho(t), && t_0 - r \le t \le t_0. \end{aligned} \tag{3.11.5}$$

PROPOSITION 3.4. *The initial value problem*

$$\begin{aligned} \dot{\boldsymbol{\varphi}}_\tau(t) &= J_\tau \boldsymbol{\varphi}_\tau(t-r), && t > t_0, \quad r > 0 \\ \boldsymbol{\varphi}_\tau(t) &= \boldsymbol{\phi}_\tau(t), && t_0 - r \le t \le t_0, \end{aligned} \tag{3.12.1}$$

where $\boldsymbol{\phi}_\tau(t)$ *is an initial function with each coordinate belonging to the set* $C([t_0 - r, t_0])$, *has a unique solution.*

The proof may be treated by the known classical methods (cf. [1, 3, 4, 8, 12, Sect. 2].

PROPOSITION 3.5. *The initial value problem*

$$\begin{aligned} H_q \dot{\boldsymbol{\varphi}}_q(t) &= \boldsymbol{\varphi}_q(t-r), && t > t_0, \quad r > 0 \\ \boldsymbol{\varphi}_q(t) &= \mathbf{0}, && t_0 - r \le t \le t_0 \end{aligned} \tag{3.13.1}$$

has only the trivial solution.

The proof may also be found in [12, Sect. 2].

From the propositions mentioned above we can state the following.

THEOREM 3.2. *The initial value problem for the homogeneous generalized linear singular delay system of the form*

$$\begin{aligned} F\dot{\mathbf{x}}(t) &= G\mathbf{x}(t-r), && t > t_0, \quad r > 0 \\ \mathbf{x}(t) &= \boldsymbol{\phi}(t), && t_0 - r \le t \le t_0 \end{aligned} \tag{3.16}$$

is solvable in $C([t_0 - r, \infty)) \cap C^1([t_0, \infty))$ *provided that the initial function* $\boldsymbol{\phi}(t)$ *has all its coordinates belonging to the set* $C([t_0 - r, t_0])$ *and has the form* $\boldsymbol{\phi}(t) = [\boldsymbol{\phi}_1^t(t)\ \boldsymbol{\phi}_2^t(t)\ \boldsymbol{\phi}_3^t(t)\ \boldsymbol{\phi}_4^t(t)]^t$, $t_0 - r \le t \le t_0$ *where* $\boldsymbol{\phi}_1(t) \in \mathbb{R}^{\nu_1 + \nu_2 + \cdots + \nu_{n-r-g} + n - r}$, $\boldsymbol{\phi}_2(t) \in \mathbb{R}^{\rho_1 + \rho_2 + \cdots + \rho_{m-r-h}}$ *with* $\boldsymbol{\phi}_2(t_0) = \mathbf{0}$, $\boldsymbol{\phi}_3(t) \in \mathbb{R}^\tau$ *and* $\boldsymbol{\phi}_4(t) \in \mathbb{R}^q$ *with* $\boldsymbol{\phi}_4(t_0) = \mathbf{0}$.

As far as uniqueness is concerned, we can state the following theorem.

THEOREM 3.3. *The initial value problem* (3.16) *has a unique solution if and only if the right nullspace of* $sF - G$ *is trivial, i.e., when the corresponding pencil of* (3.16) *has no non-zero column minimal indices.*

The proof is a direct consequence of Proposition 3.2 and Theorem 3.2.

REFERENCES

1. R. Bellman and K. L. Cooke, “Differential–Difference Equations,” Academic Press, New York, 1963.

2. S. L. Campbell, "Singular Systems of Differential Equations," Vols. 1 and 2, Pitman, San Francisco, 1980, 1982.
3. R. D. Driver, "Ordinary and Delay Differential Equations," Springer-Verlag, New York, 1977.
4. L. E. Elsgolts, "Introduction to the Theory of Differential Equations with Deviating Arguments," Holden–Day, San Francisco, 1966.
5. P. Van Dooren, Reducing subspaces: definitions, properties, and algorithms, *in* "Matrix Pencils," Lecture Notes in Math., Vol. 973, Springer-Verlag, Berlin/New York, 1983.
6. R. F. Gantmacher, "The Theory of Matrices," Chelsea, New York, 1977.
7. E. Grispos, Singular generalized autonomous linear differential systems, *Bull. Greek. Math. Soc.* **34** (1992), 25–43.
8. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.
9. G. Kalogeropoulos, "Matrix Pencils and Linear Systems," Ph.D. thesis, City University, London, 1985.
10. N. Karcanias and G. E. Hayton, Generalised autonomous differential systems, algebraic duality, and geometric theory, *in* "Proceedings of IFAC VIII, Triennial World Congress, Kyoto, Japan 1981."
11. N. Karcanias and B. Kouvaritakis, The output zeroing problem and its relationship to the invariant zero structure, *Internat. J. Control* **30** (1979), 395–415.
12. G. Kalogeropoulos and I. G. Stratis, On generalized linear regular delay systems, *J. Math. Anal. Appl.* **237** (1999), 505–514.
13. G. Birkhoff and S. Maclane, "A Survey of Modern Algebra," Macmillan, New York, 1965.
14. D. G. Forney, Minimal bases of rational vector spaces with application to multivariable systems, *SIAM J. Control* **13** (1975), 493–520.